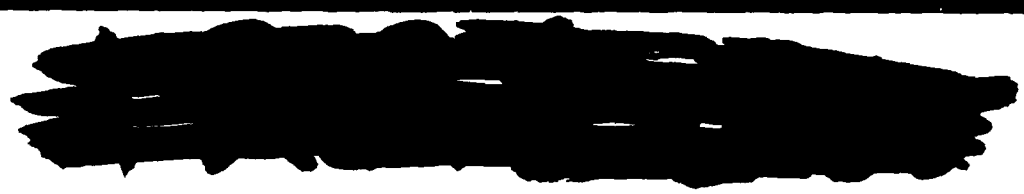


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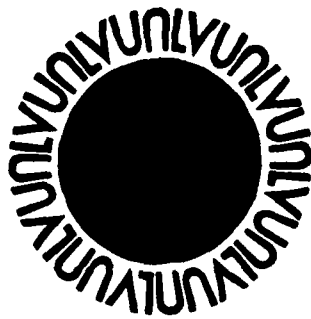


**External Watchman Routes**

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**Abstract:** We consider the problem of finding a shortest watchman route from which the exterior of a polygon is visible (external watchman route). We present an  $O(n^4 \log \log n)$  algorithm to find shortest external watchman routes for simple polygons by transforming the external watchman route problem to a set of internal watchman route problems. Also, we present faster external watchman route algorithms for special cases. These include optimal  $O(n)$  algorithms for convex, monotone, star and spiral polygons and an  $O(n \log \log n)$  algorithm for rectilinear polygons.

**Key Words:** Watchman Routes, Visibility, Art Gallery Problem.

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## I. Introduction.

Path planning and visibility are two central areas in Computational Geometry and Robotics. In path planning one usually wants to find a shortest collision-free path between two points. In many cases there is a close relation between shortest paths and visibility. For example, 2-d shortest paths are found using visibility graphs [LP84, We85, GM87]; algorithms for shortest paths inside a polygon make heavy use of visibility properties [AA\*85, GH\*86]. Most research on visibility deals with stationary vision. The Art Gallery problem asks for the minimum number of watchmen (guards) that need to be stationed in a polygonal Gallery so that each point in the Gallery is visible to at least one of them. There has been considerable research on the Art Gallery problem and its variations, most of which can be found in [O'R87]. Some visibility problems from a moving viewpoint are considered in [EO83].

Watchman routes combine path planning and visibility considerations. The watchman route problem [CN88] asks for the shortest route from a point  $s$  back to itself and with the property that each point in a given space is visible from at least one point along the route. Finding shortest watchman routes in polygons with holes and simple polyhedra is NP-Hard [CN88]. An  $O(n \log \log n)$  algorithm for finding shortest watchman routes in simple rectilinear polygons is given in [CN88] and an  $O(n^4 \log \log n)$  algorithm for simple polygons is presented in [CN87]. These algorithms identify a set of segments inside the polygon that a watchman route must visit and construct a shortest route by unrolling the polygon (using the segments as mirrors) and solving a shortest path problem in the resulting polygon.

While most research deals with shortest paths and visibility issues for the interior of polygons, there has been significant interest on similar problems for the exterior. The Art

Gallery problem, when translated to the exterior of the polygon, becomes the Fortress problem [O'R87]. Weak internal and external visibility are considered in [AT81, TA82]. Sector visibility, a restricted form of weak visibility, is considered in [BKT89]. The problem of computing the external geodesic diameter of a simple polygon is considered in [ST87, AA\*89].

In this paper we look at the external watchman route problem. We are given a simple polygon and we want to design a shortest route such that each point in the exterior of the polygon is visible from some point along the route. There are many similarities between the external and internal watchman route problems but also significant differences. The external watchman route problem is at least as hard as the internal one because we can use the following reduction to obtain a shortest internal watchman route by solving an external watchman route problem. Given a polygon  $Q$  and a starting point  $s$  in its boundary, we enclose  $Q$  in a rectangle and connect one of the sides of the rectangle to the starting point  $s$  with a thin corridor (see Figure 1). Then, any shortest external route in the resulting polygon  $Q'$  will contain the shortest internal route as a subpath. Note that the above transformation of the internal to the external watchman route problem takes  $O(n)$  and assumes that the starting point  $s$  on the boundary of the polygon is given as part of the input. The complexity of the internal watchman route problem for simple polygons when a starting point is not specified remains an open problem.

In the next section we present an optimal  $O(n)$  time algorithm for finding shortest external watchman routes for convex polygons. In Section III we present an  $O(n^4 \log \log n)$  algorithm to find a shortest external route for a simple polygon. In section IV we consider external routes for special classes of polygons.

## II. External Watchman Routes for Convex Polygons.

Let  $Q$  be a convex polygon with  $n$  vertices  $v_1, \dots, v_n$  indexed as they are visited in a clockwise scan of the boundary of  $Q$ . The edges of  $Q$  are indexed similarly, i.e.,  $e_i = (v_{i-1}, v_i)$ ,  $1 \leq i \leq n$  (with  $v_0 = v_n$ ). We want to find a shortest route from which the exterior of  $Q$  is visible.

Since  $Q$  is convex, its convex hull is the same as the boundary of  $Q$ . Note that the route that follows the boundary of  $Q$  is an external watchman route. In fact it may be a shortest external watchman route. We refer to external routes that wrap all the way around the polygon as **convex hull routes**, i.e., convex hull routes have the property that any outgoing ray from any point on the boundary of  $Q$  intersects the route. We refer to all other external watchman routes as **2-leg routes**. A route  $\pi$  of this type has two **extreme points** (e.g., points  $x, y$  in Figure 2) and consists of an **inner path**  $\pi_I$  and an **outer path**  $\pi_O$ , both of which connect these two points. The inner path does all the work, i.e., the exterior of  $Q$  is visible from  $\pi_I$ . The outer path serves only to complete the route. The extreme points are taken to be such that the inner path is as short as possible, i.e., we can define the extreme points to be the endpoints of the shortest subpath from which the exterior of  $Q$  is visible.

**Lemma 1:** Every shortest external watchman route must contact the boundary of  $Q$ .

**Proof:** Let  $\pi$  be a shortest external watchman route for  $Q$  and suppose that  $\pi$  does not contact the boundary of  $Q$ . If  $\pi$  is a convex hull route, construct a supporting line of  $Q$  and find its first intersections with  $\pi$  on either side, say at points  $x, y$ . Let  $\pi'$  be the convex hull route obtained by replacing one of the spans connecting  $x, y$  with the supporting segment between these points. If  $\pi$  is a 2-leg route there must be some point  $z$  along the boundary of  $Q$  from which some outgoing ray does not intersect  $\pi$ . Construct the two supporting lines of

$\pi$  from the point  $z$  on either side of this ray and let  $x, y$  be contacts of the supporting lines with  $\pi$  respectively. Rotate  $\pi$  about  $x$  so that  $\pi$  will intersect the supporting line through  $y$  until  $\pi$  contacts the boundary of  $Q$ . Let  $\pi'$  be the 2-leg route obtained from  $\pi$  by rotating as described above and replacing the portion(s) of  $\pi$  that has rotated past the supporting line through  $y$  with the corresponding segment(s) of the supporting line. In both cases  $\pi'$  is still an external watchman route and is shorter than  $\pi$ , a contradiction. **Q.E.D.**

Then, without loss of generality, we only consider routes that contact  $Q$ . The inner path of a 2-leg route can be further partitioned into three subpaths. The middle subpath  $\pi_M$  is the **body** of the route and, for convex polygons, it follows the boundary of  $Q$ ; the two outside sections  $\pi_L, \pi_R$  are the **left, right legs** of the route and connect  $\pi_M$  with the extreme points  $x, y$ . The legs start at the vertices where  $\pi_I$  first leaves the boundary of  $Q$  on either side. Note that any one of the three subpaths (and even both legs) of  $\pi_I$  may be empty.

**Theorem 1:** A shortest external watchman route  $\pi$  for a convex polygon  $Q$  either follows the boundary of  $Q$  or it is a 2-leg route in which the two legs are supporting segments of  $Q$  that are perpendicular to the extensions of two adjacent edges of  $Q$ ,  $\pi_M$  follows the boundary of  $Q$  and  $\pi_O = \pi_I$ .

**Proof:** First consider the class of convex hull routes. The route that follows the boundary is the shortest in that class. Of the remaining routes, suppose that there is a route  $p$  that is not a 2-leg route of the type described above but is shorter than a shortest 2-leg route  $\pi$ . There is a pair of adjacent edges in  $Q$  such that each leg of  $p$  comes in contact with the extension of one of these edges (if not,  $p$  does not see one of the edges and can not be an external watchman route). But then the route  $\pi'$  (of the type described in the statement of the Theorem) with legs that are supporting segments of  $Q$  and perpendicular to the exten-

sions of the same two edges is not longer than  $p$  and, since  $\pi$  is the shortest among all the 2-leg routes like  $\pi'$ , it follows that  $p$  is at least as long as  $\pi$ , a contradiction. **Q.E.D.**

To construct a shortest external watchman route, we consider all pairs of adjacent edges  $e_i, e_{i+1}$  and construct the shortest 2-leg route for each pair (each route will have legs perpendicular to the corresponding pair of edges). Then we select the shortest of the 2-leg routes, compare it with the route that follows the boundary of  $Q$  and report the shorter of the two as a shortest external watchman route.

**Theorem 2:** A shortest external watchman route for a convex polygon can be constructed in  $O(n)$ .

**Proof:** For a given pair of adjacent edges the construction of the shortest 2-leg route takes  $O(n)$ . The perpendicular supporting segments can be found in  $O(\log n)$  using binary search on the boundary of the polygon (similar to supporting line algorithms [PS85]). Having one 2-leg route, we can find the rest of them in  $O(n)$  by walking around the polygon. As we consider successive pairs of adjacent edges, the contacts of the perpendicular supporting segments advance around the polygon and all the 2-leg routes can be computed at a cost of  $O(1)$  per edge since we consider each edge a constant number of times during these advances. Computing the length of the boundary route can be done in  $O(n)$ . Then the overall complexity is  $O(n)$ . Note that this is optimal since it may take  $O(n)$  to report a shortest route. **Q.E.D.**

The example of Figure 2 establishes that there can be at least one 2-leg route that is shorter than any convex hull route. A vertex  $v$  of  $Q$  is a **convex separator**, if the corresponding two-leg route (with legs perpendicular to the extensions of the edges incident on  $v$ ) is shorter than the shortest convex hull route. It is useful to determine how many con-

vex separators a convex polygon  $Q$  can have. We use the notation  $x-y$  to refer to the portion of the convex hull of the polygon from point  $x$  to point  $y$  in a clockwise scan and  $[x-y]$  to represent its length. Intermediate points may be included for clarity. For a straight line segment connecting  $x, y$ , we use  $xy$  to represent the segment and  $[xy]$  to represent its length.

Let  $\pi$  be a 2-leg route that is shorter than the shortest convex hull route, let  $v$  be the corresponding convex separator and let  $a, b$  be the vertices where the two legs of  $\pi$  attach to the polygon. The body of  $\pi$  is the span  $a-b$  and all other vertices in this span are said to be in the interior of the body of  $\pi$ .

**Lemma 2:** If  $v'$  is a convex separator for  $Q$  then  $v'$  can not be in the interior of the body of  $\pi$ .

**Proof:** Let  $\pi'$  be the 2-leg route corresponding to  $v'$  and suppose that  $v'$  lies in the interior of the body of  $\pi$ . Then the two routes must cross each other on both sides of the polygon; otherwise  $\pi'$  will not be a watchman route as the edges incident on  $v'$  would not be visible from it (Figure 3 illustrates a typical case). But then we have that the union of portions of the inner paths of the two routes wraps completely around  $Q$ , i.e. it is at least as long as the shortest convex hull route. For example, in Figure 3, the two routes cross at points  $u, t$  and the union of the portions  $[u-a-v'-b-t]$ ,  $[t-a'-v-b'-u]$  of the inner paths of the two routes is longer than the shortest convex hull route. Since the outer paths of  $\pi, \pi'$  are equal in length to the inner ones, it is impossible for both  $\pi$  and  $\pi'$  to be shorter than the shortest convex hull route. Thus  $v'$  can not lie in the interior of the body of  $\pi$ . **Q.E.D.**

We define the **angle** of a convex separator to be the interior angle in  $Q$  that is formed by the edges incident on the separator (i.e., angles are always less than 180 degrees because

of the convexity of  $Q$ ). Let  $v$  be the convex separator with maximum angle in  $Q$  (arbitrarily break any ties). From Lemma 2 we have that there is no other convex separator in the interior of the body of  $\pi$ , the 2-leg route corresponding to  $v$ . Consider now additional convex separators that may lie outside the body of  $\pi$ .

**Lemma 3:** There can not be a second convex separator for  $Q$  that lies outside the body of  $\pi$ .

**Proof:** Without loss of generality, assume that there exists a second convex separator  $v'$  in the span  $b-v$  (see Figure 4a; the case where  $v'$  is in the span  $v-a$  is similar). Note that the convexity of the polygon implies that  $a'$  is at or counterclockwise from  $a$  and  $b'$  is at or counterclockwise from  $b$ . Also, vertex  $b'$  can not be in the span  $v'-v$  (we can not have a perpendicular supporting line from it to an edge incident on  $v'$ ) and vertex  $a'$  can not be in the interior of the span  $v-v'$  (that would place  $v$  in the interior of the body of the route corresponding to  $v'$  contradicting Lemma 2).

Consider the case shown in Figure 4a. If both  $\pi, \pi'$  are shorter than the shortest convex hull route, we have:

$$2([ya'] + [a'-a-b'] + [b'x]) < [v'-a'-b'-v'],$$

$$2([wa] + [a-b'-b] + [bz]) < [v-a-b-v] = [v'-a'-b'-v'].$$

If we take the sum of the two inequalities, remove common terms and divide by two we have:

$$[b'x] + [ya'] + [bz] + [wa] + [ab'] < [b-v'-v-a'].$$

But  $[b-v'-v-a']$  is less than  $[bz] + [zy] + [ya']$  and we have:

$$([wa] + [ab'] + [b'x]) < [zy].$$

Since the polygon is convex, the lines through  $bz, a'y$  intersect below the polygon (or are

parallel if  $v, v'$  are adjacent). Then  $[wa] + [ab'] + [b'x]$  is larger than  $[zy]$  (see detail in Figure 4a), a contradiction.

Similar arguments apply to all cases where  $w, x$  are both at or outside the lines through  $ya'$  and  $bz$ . Since  $v' \neq b$ , point  $x$  is always outside the line through  $bz$ . However, point  $w$  may be inside the line through  $ya'$  and this occurs only if  $a' = v$ . For these cases, we show that  $v'$  has a larger angle than  $v$ , contradicting our assumption that  $v$  is a separator with largest angle. This situation is illustrated in Figure 4b where we have that  $a = b'$  (the case where  $a \neq b'$  is similar). Let  $\theta$  be the angle for  $v$  and let  $\theta'$  be the angle for  $v'$ . Angle  $\theta'$  is always greater than 90 degrees because  $v'$  is inside the right angle formed by  $bz$  and the extension of the edge incident on  $v$ . Since  $v, v'$  are convex separators, we have that:

$$2([wa] + [a-b] + [bz]) < [v-a-b-v]$$

$$2([yv] + [v-b'] + [b'x]) < [v'-v-b'-v']$$

If we take the sum, remove common terms and divide by two we have:

$$([wa] + [b'x]) < [yz] < [vz]$$

Consider now the quadrilateral shown in Figure 4b (detail) with corners  $v, w', x$  and  $z'$ . The angles at corners  $w', z'$  are right angles. Then, from the relation above, it follows that the side  $w'x$  is smaller than the side  $vz'$ . This implies that the lines through  $vw'$  and  $z'x$  intersect above the quadrilateral which means that the angle at corner  $v$  is less than 90 degrees. This contradicts our assumption that  $v$  is a convex separator with largest angle (we have already shown that  $v'$  has an angle larger than 90 degrees). Similar arguments hold for the case when  $b'$  is clockwise from  $a$  or  $b'$  is in the interior of the span  $v-a$ . Thus, there can not be a second convex separator outside the body of the 2-leg route corresponding to the convex separator with largest angle. **Q.E.D.**

From the above lemmas it follows that, if  $v$  is a convex separator with largest angle in  $Q$ , there can not exist a second convex separator in the interior or outside of the body of the corresponding 2-leg route  $\pi$ . Then the only vertices where additional convex separators may occur are the vertices  $a, b$ , the endpoints of the body of  $\pi$ .

**Corollary 1:** A convex polygon  $Q$  can have at most three convex separators.

A triangle is an example of a convex polygon that has three convex separators.

### III. External Watchman Routes in Simple Polygons.

Let  $Q$  be a simple polygon with  $n$  vertices  $v_1, \dots, v_n$  and  $n$  edges  $e_i = (v_{i-1}, v_i)$ ,  $1 \leq i \leq n$  (with  $v_0 = v_n$ ). Consider the extensions of each polygon edge to the exterior of the polygon. Each of them borders a region (bounded by the half-plane associated with the edge and possibly the boundary of the polygon and other half-planes) from which the corresponding edge is visible. An external watchman route may see an edge either by crossing its extension, or by visiting a point in the region from which the whole edge is visible or by coming up to its extension and turning back (Figure 5). We define a **cave** as any area between the polygon and the convex hull such that there is at least one edge in it such that the region from which it is visible does not intersect the convex hull of the polygon. The areas enclosed by the convex hull edges  $e_1, e_2$  in Figure 6 are caves. The significance of caves lies in the fact that any external route must enter a cave in order to see the edges inside it, i.e., patrolling along the convex hull is not sufficient.

As in the convex case, we have two types of routes for a simple polygon. The shortest **convex hull route** follows the convex hull of  $Q$  entering caves along the way. A shortest convex hull route is shown in Figure 6. Again we have **2-leg routes** but there is a greater variety of them that can be shortest external watchman routes overall. Unlike the convex

case, a shortest 2-leg route does not necessarily follow a path twice. The body of its inner path may need to enter caves along the way while the outer path will just follow the convex hull in those areas. We let the legs start at the vertices in the convex hull where the outer path last goes outside the convex hull on either side of the body of the route. The various types of legs that occur in shortest 2-leg routes are illustrated in Figure 7. Each leg of the route can be a supporting segment that is perpendicular to the extension of some edge (Figure 7a), a supporting segment from the intersection of two such extensions (Figure 7b), a supporting segment from an endpoint of a polygon edge (Figure 7c; note that the leg is empty in this case) or a full fledged watchman path that enters a cave (Figure 7d) or visits a cave and the extensions of some additional edges (Figure 7e). In the first three cases (Figures 7a-c) the outer path will overlap with the legs and follow the convex hull in the section covered by the body of the route.

**Lemma 4:** The shortest convex hull route can be constructed in  $O(n^4 \log \log n)$ .

**Proof:** The shortest convex hull route follows the convex hull except in areas where a convex hull edge  $(v_i, v_j)$  encloses a cave. We need to find a shortest watchman path from  $v_i$  to  $v_j$  from which the interior of the cave (i.e., the corresponding exterior of the polygon) is visible. This problem is equivalent to the internal watchman route problem in a simple polygon (the cave) and the algorithm in [CN87] can be used to find a shortest path. (The fact that the watchman path for a cave has a fixed starting point and a fixed, but different, end point is not significant; it requires a minor and trivial modification to the algorithm in [CN87]). **Q.E.D.**

To compute the shortest 2-leg route for a simple polygon we need a way to separate the edges of the polygon into three sets; two of the sets will contain the edges visible to each of

the legs while the third set will contain the remaining edges (these will be seen from the body of the route). If we know the edges that each leg is to cover, we can construct a shortest 2-leg route by solving an internal watchman route problem. However there is an exponential number of ways in which to partition the edges of  $Q$  into the three sets. The number of partitions we need to consider is significantly reduced by the following observation.

**Lemma 5:** If a 2-leg route is shorter than the shortest convex hull route, then the extreme points in each leg can not be visible to each other.

**Proof:** If the extreme points are visible to each other, the straight line segment connecting them is shorter than the outer path of the 2-leg route. Therefore the 2-leg route can not be shorter than the shortest convex hull route. **Q.E.D.**

Lemma 5 establishes that there must be a spike in the boundary of the polygon between the legs of a 2-leg route. The spike is large enough so that it is preferable for the route to go back around the rest of the polygon rather than go around the spike. Note that, in addition to the extreme points in the two legs, Lemma 5 holds for any pair of points in the outer path that are not in the same segment. In addition, we can not have an edge to the left (right) of the spike covered by the leg that is to the right (left) of the spike.

**Lemma 6:** Let  $\pi$  be a 2-leg route that is shorter than the shortest convex hull route for a simple polygon  $Q$  and let  $a, b$  be the convex hull vertices where the left, right legs start respectively. Then there is a vertex  $v$  on the convex hull of  $Q$  such that all edges in the span  $v-b$  are seen by the right leg and all edges in the span  $a-v$  are seen by the left leg of the route.

**Proof:** From Lemma 5 we have that there is a vertex  $v$  on the convex hull of  $Q$  that lies between the two legs of the route  $\pi$ . Suppose that there is an edge  $e$  in the span  $v-b$  that is seen by the left leg of the route (the leg that lies on the opposite side of  $v$ ) and an edge  $e'$  between  $v$  and  $e$  that is seen by the right leg of the route (see Figure 8). (Note that, if this situation does not occur for  $v$ , some other vertex in the span  $v-b$  will satisfy the conditions of the Lemma). Since edges  $e, e'$  are at or inside the convex hull of the polygon, the convexity of the convex hull implies that the two legs are visible to each other contradicting Lemma 5. **Q.E.D.**

Let  $v$  be a vertex that satisfies the conditions of Lemma 6. Vertex  $v$  defines a shortest 2-leg route since it determines the partition of the edges for the two legs of the route. Since  $v$  is a convex hull vertex, it follows that we need consider at most  $O(n)$  2-leg routes. If the shortest 2-leg route corresponding to  $v$  is shorter than the shortest convex hull route for  $Q$ ,  $v$  is a **separator** of  $Q$ . We can reduce the number of 2-leg routes that we need to construct by making use of Corollary 1 for convex polygons to show that the number of separators of  $Q$  is small.

**Lemma 7:** Any separator for  $Q$  is also a convex separator for the convex hull of  $Q$ .

**Proof:** Let  $v$  be a separator of  $Q$  and consider the corresponding shortest 2-leg routes  $\pi, \rho$  for  $Q$  and for the convex hull of  $Q$  respectively. Let  $a, b$  be the vertices of the convex hull where the legs of  $\rho$  start. Let  $x, y$  be the extreme points of  $\rho$  (see Figure 9). In order for  $\rho$  to be shorter than the convex hull of  $Q$ , we should have:

$$[ax] + [x-a-b-y] + [yb] < [h-v-a] \quad (1)$$

To distinguish between paths along the convex hull and paths along the 2-leg route  $\pi$ , we use the notation  $a \equiv b$ ,  $[a \equiv b]$  for the latter. Let  $X, Y$  be the intersections of  $\pi_0$  with the

convex hull that are closest to the extreme points of  $\pi$ : Since  $\pi$  is shorter than the shortest convex hull route for  $Q$ , we have:

$$[a \Rightarrow X] + [X - a - b - Y] + [Y \Rightarrow h] < [b \Rightarrow v \Rightarrow a]$$

But  $[b \Rightarrow v \Rightarrow a]$  is less than  $[b \Rightarrow Y] + [Yv] + [vX] + [X \Rightarrow a]$ . If we replace the right hand side above we have that  $[X - a - b - Y] < [Yv] + [vX]$  or equivalently:

$$[X - a - b - Y] + [X - a] + [b - Y] < [b - Y] + [Yv] + [vX] + [X - a]. \quad (2)$$

Comparing inequalities (1), (2) we have that the left hand side of (2) is larger than the left hand side of (1) and the right hand side of (1) is larger than the right hand side of (2). That is, inequality (2) is harder to satisfy than inequality (1). Thus, inequality (1) holds and  $v$  is a convex separator for the convex hull of  $Q$ . **Q.E.D.**

**Theorem 3:** A shortest external watchman route for a simple polygon can be constructed in  $O(n^4 \log \log n)$ .

**Proof:** From lemma 7 and corollary 1, it follows that there are at most three separators to consider in constructing 2-leg routes. Moreover, these separators can be found in  $O(n)$  time. Let  $v$  be such a separator and let  $u$  be a convex hull vertex that is contacted by the shortest 2-leg route for the convex hull of  $Q$ . We enclose the polygon in a large rectangle and connect  $v$  to the rectangle with a corridor that is perpendicular to the side opposite  $v$ . Then a shortest 2-leg route can be constructed by solving the internal shortest watchman route problem in the resulting polygon with  $u$  as the starting point of the route. This can be done in  $O(n^4 \log \log n)$  [CN87]. We repeat this for any additional separators. Since there are at most three separators, the complexity remains  $O(n^4 \log \log n)$ . From Lemma 4, we have that the shortest convex hull route for  $Q$  can be constructed in  $O(n^4 \log \log n)$  as well. Then the shorter of the shortest convex hull route and the up to three 2-leg routes is a shortest

external watchman route for  $Q$ .

Note that the complexity of the external watchman route problem would be reduced if faster algorithms for the internal watchman route problem were available. Essentially, we are reducing the external watchman route problem into up to three instances of the internal watchman route problem on polygons of similar size (to find the 2-leg routes associated with the separators) and a set of smaller internal watchman route problems for the caves. The complexity of the transformation itself is  $O(n \log n)$  since we need to construct the convex hull to identify the caves (the rest takes only  $O(n)$ ). **Q.E.D.**

#### IV. External Routes for Restricted Classes of Polygons

In the previous section we presented an  $O(n^4 \log \log n)$  algorithm to find a shortest external watchman route for a simple polygon. The main contributing factor to this complexity is the complexity of the internal watchman route problem. There are special classes of polygons for which internal routes can be found faster. An  $O(n)$  algorithm for monotone rectilinear polygons and an  $O(n \log \log n)$  algorithm for rectilinear polygons are presented in [CN88]. From Section II, we have that special classes of polygons may allow the shortest external route to be constructed directly (without solving an internal route problem). In this section we develop faster algorithms for some additional classes of polygons.

**Rectilinear Polygons:** In a rectilinear polygon, the caves will also be rectilinear. Then we can identify a separator (if there is one) in  $O(n)$  using the convex hull of the polygon and we can construct the shortest 2-leg route and the shortest convex hull route in  $O(n \log \log n)$  using the internal watchman route algorithm in [CN88]. Thus we have:

**Corollary 2:** A shortest external watchman route for a rectilinear polygon can be found in  $O(n \log \log n)$ .

**Monotone Polygons:** A polygon is monotone if its boundary consists of two chains that are monotone with respect to an axis of monotonicity. Without loss of generality we assume that the polygon is monotone with respect to a vertical line. A monotone polygon can not have any caves (the vertices in a cave do not project monotonically on any axis of monotonicity for the convex hull). The shortest convex hull route is then the convex hull itself and can be constructed in  $O(n)$ . If there is a separator, the construction of the shortest 2-leg route involves the construction of extreme chains on either side of each separator (Figure 10). Each extreme chain is actually a portion of the boundary of the region (kernel) from which the edges on one side of the separator are visible. To construct each chain in  $O(n)$  we start from the edges adjacent to the separator and build up each chain as we go upward. For each new edge that is considered we use the far segment of the current chain to determine if the chain needs to be updated. If it does, we go inward on the current chain throwing away segments until we find the place where the segment corresponding to the new edge needs to be inserted (this will now become the far segment of the new chain). The total work per edge is  $O(1)$  because the cost of walking inward on a chain can be distributed on the segments that are removed. Each leg of the 2-leg route is either perpendicular to a segment in a chain or a supporting segment from one of the vertices in the chain. The latter occurs only if there is no perpendicular supporting segment. We can construct each leg in  $O(n)$  by walking inward on each chain and updating the supporting segments. Then the shortest 2-leg route can be constructed in  $O(n)$ .

**Corollary 3:** A shortest external watchman route for a monotone polygon can be constructed in  $O(n)$ .

**Star Polygons:** A star polygon has the property that there is at least one point in its interior from which all of the interior is visible. Again it is easy to see that a star polygon can not have any caves. In star polygons that have a separator, the shortest 2-leg route can be constructed as for the monotone case. Typical shortest 2-leg routes for star polygons are shown in Figure 11.

**Corollary 4:** A shortest external watchman route for a star polygon can be constructed in  $O(n)$ .

**Spiral Polygons:** A spiral polygon consists of a convex and a reflex chain. Shortest internal watchman routes for a spiral polygon can be constructed in  $O(n)$  [NW89]. With respect to external routes, a spiral polygon is essentially a convex polygon with a single spiral cave.

**Corollary 5:** A shortest external watchman route for a spiral polygon can be constructed in  $O(n)$ .

## V. Concluding Remarks.

We presented an  $O(n^4 \log \log n)$  algorithm to find a shortest external watchman route for a simple polygon and more efficient algorithms for finding shortest external routes for restricted classes of polygons. An interesting open problem is to find shortest external routes (or good approximation algorithms) for a collection of polygons. The general problem for any number of polygons is NP-hard (since finding shortest internal watchman routes in polygons with holes is intractable [CN88]).

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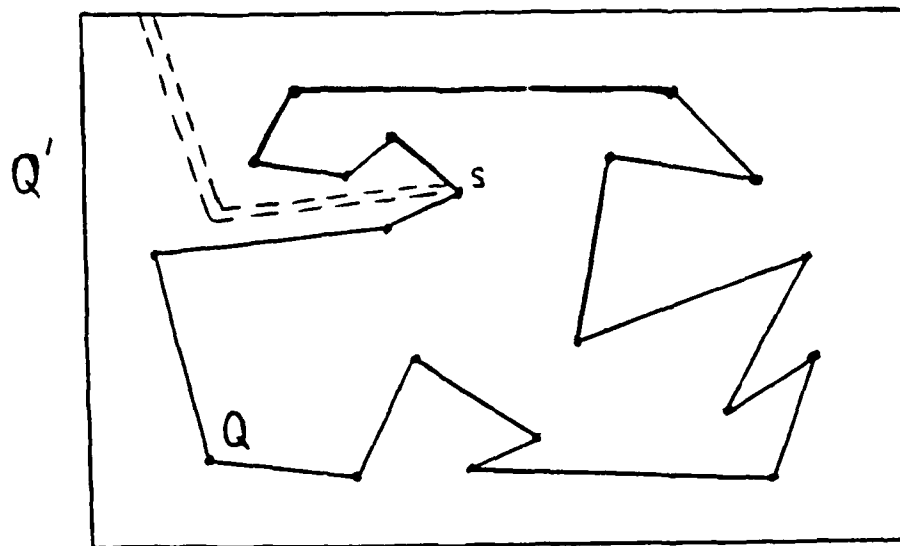


Figure 1. Reducing the internal watchman route problem to the external watchman route problem.

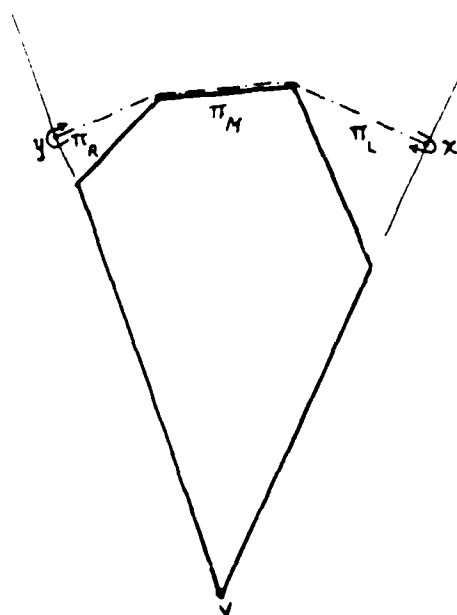


Figure 2. A 2-leg watchman route.

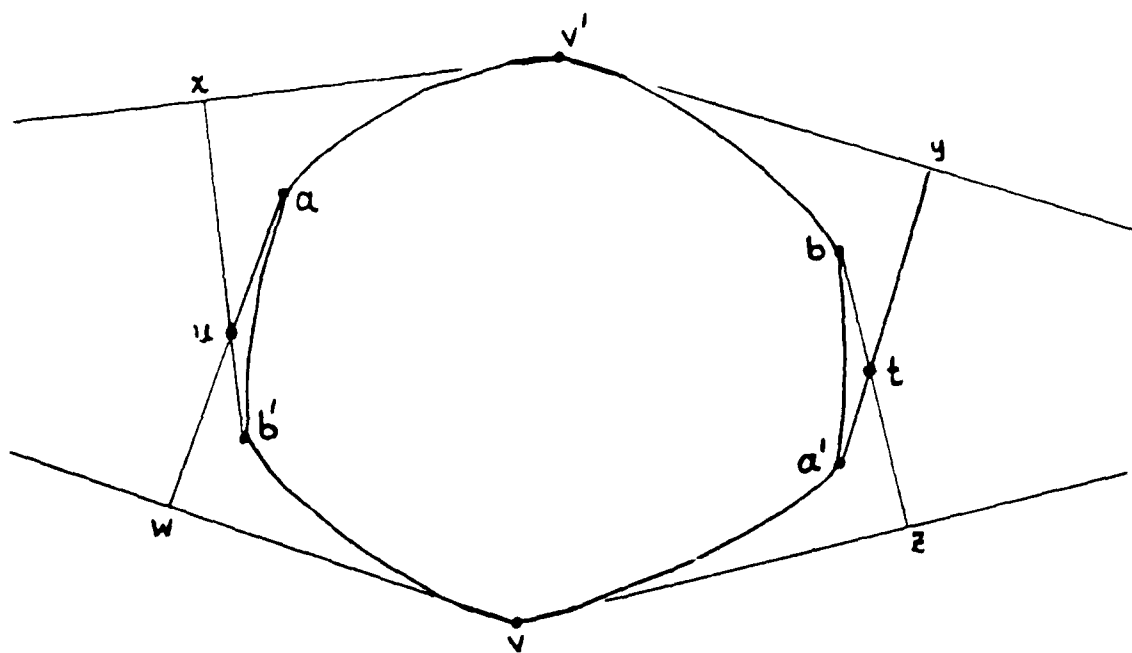
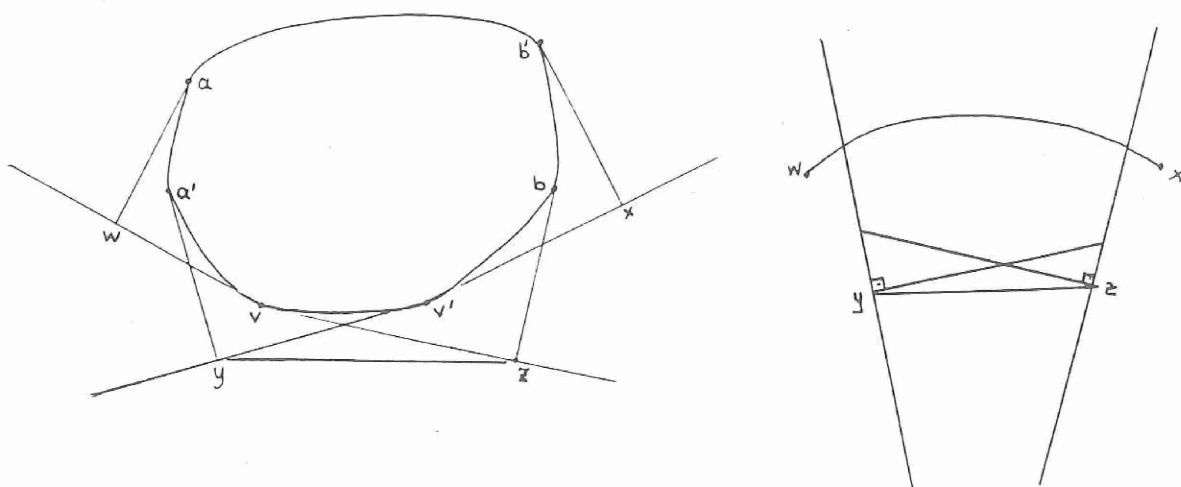
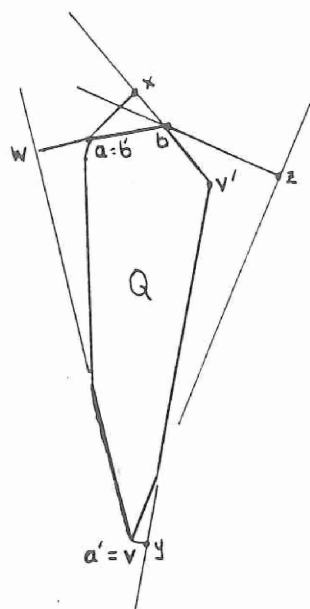


Figure 3. Proof of Lemma 2.



(a)



(b)

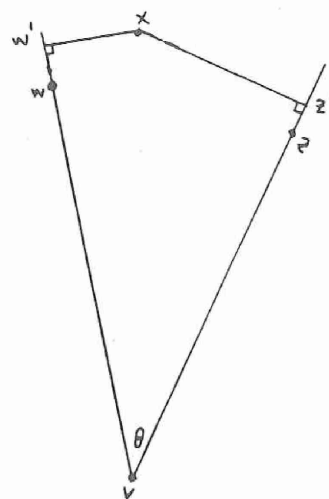


Figure 4. Proof of Lemma 3.

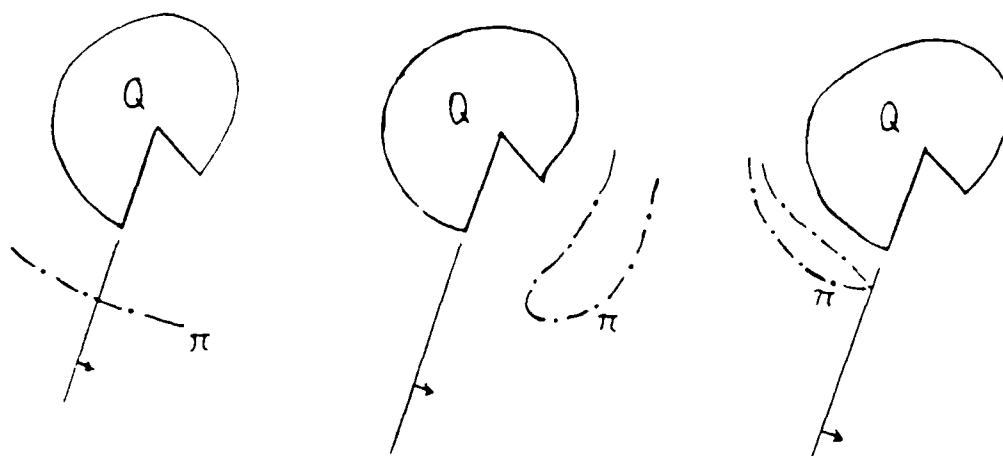


Figure 5. Various ways for an external route to see an edge.

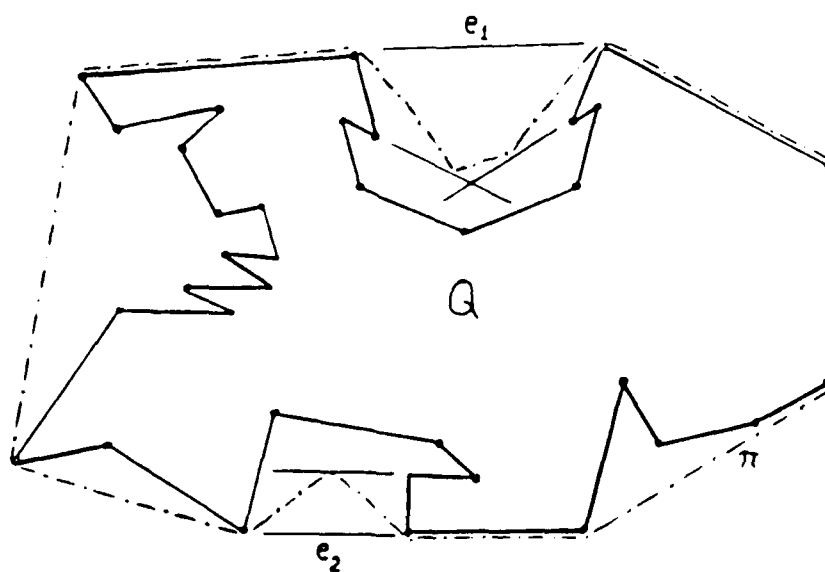
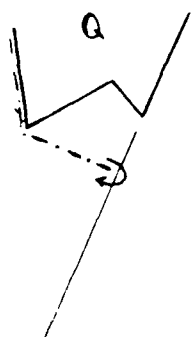
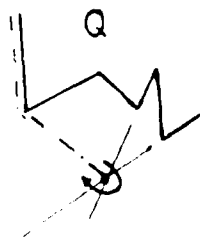


Figure 6. A shortest convex hull route.



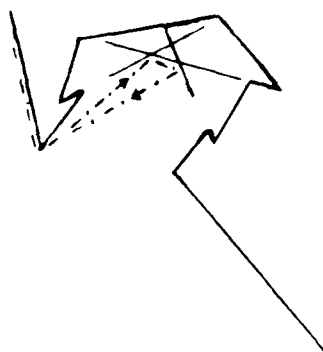
(a)



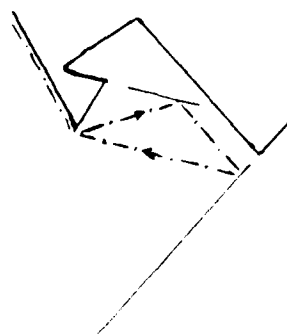
(b)



(c)



(d)



(e)

Figure 7. Types of legs for shortest 2-leg routes for simple polygons.

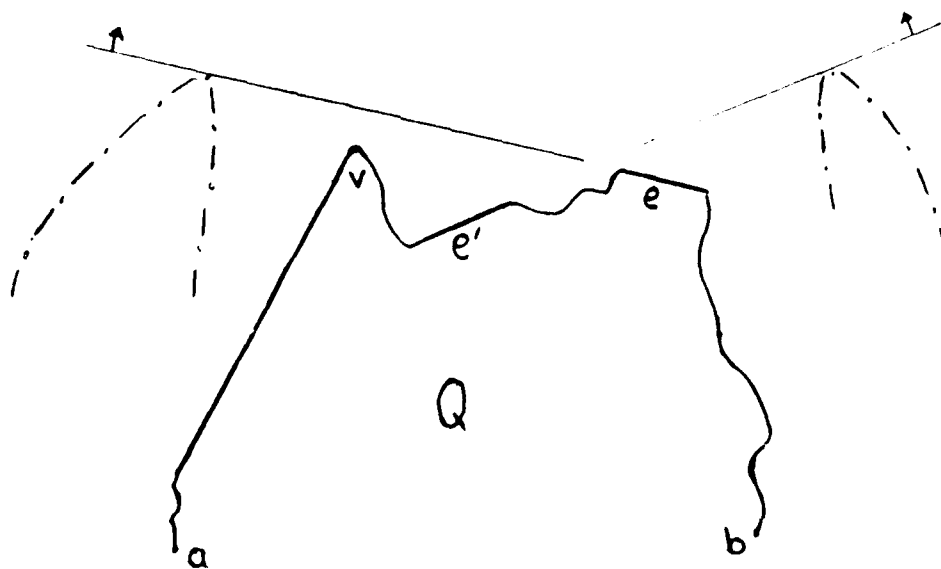


Figure 8. Proof of Lemma 6.

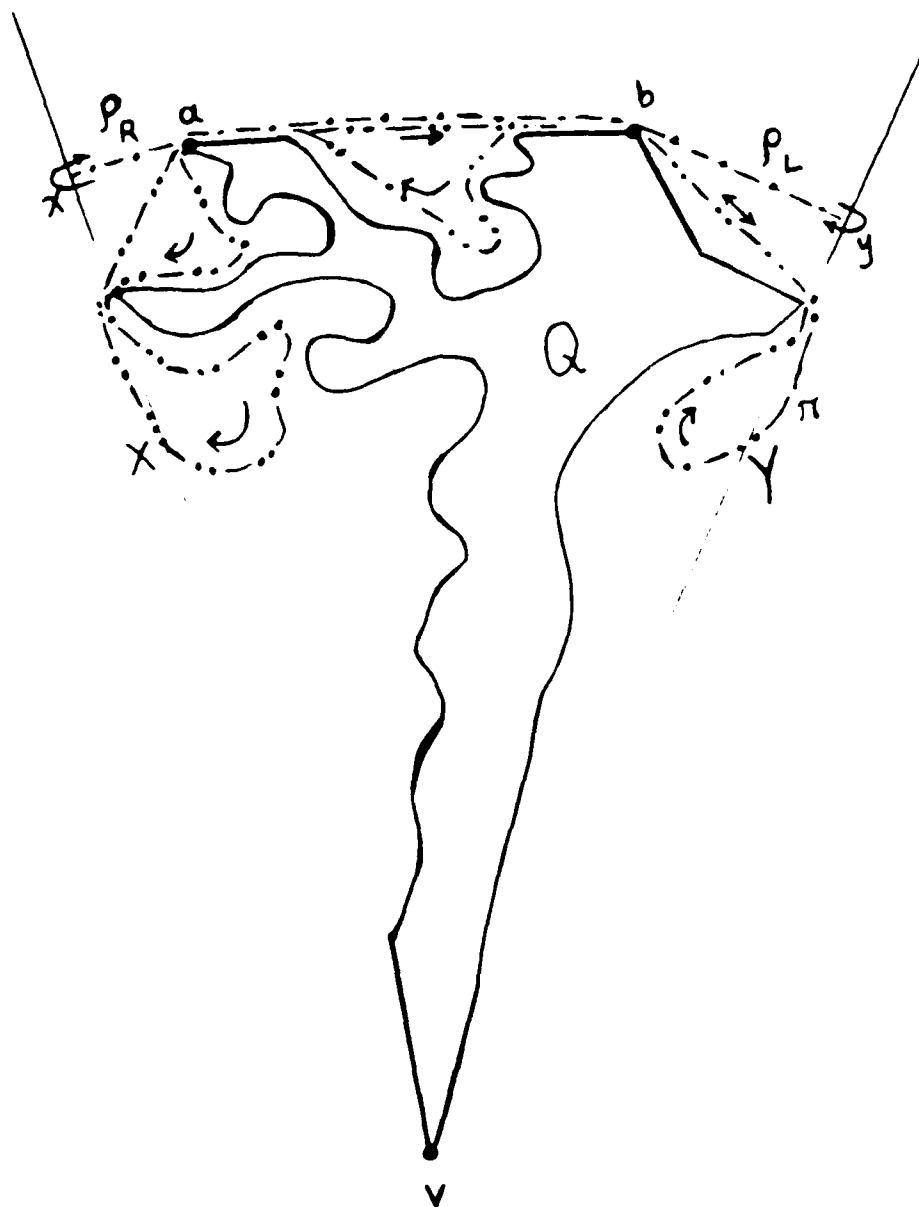


Figure 9. Proof of Lemma 7.

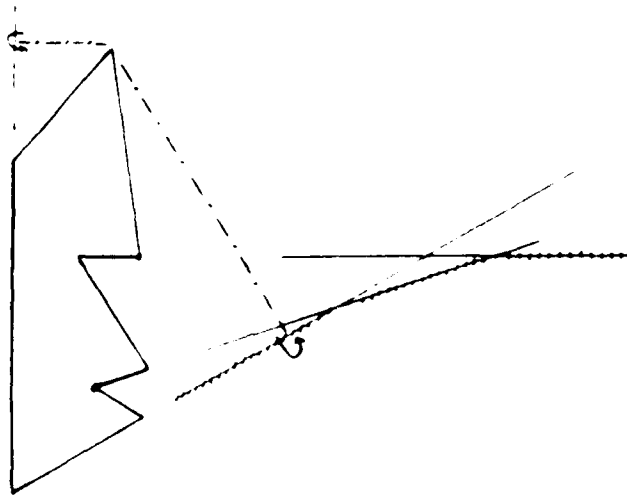


Figure 10. A shortest external route for a monotone polygon.

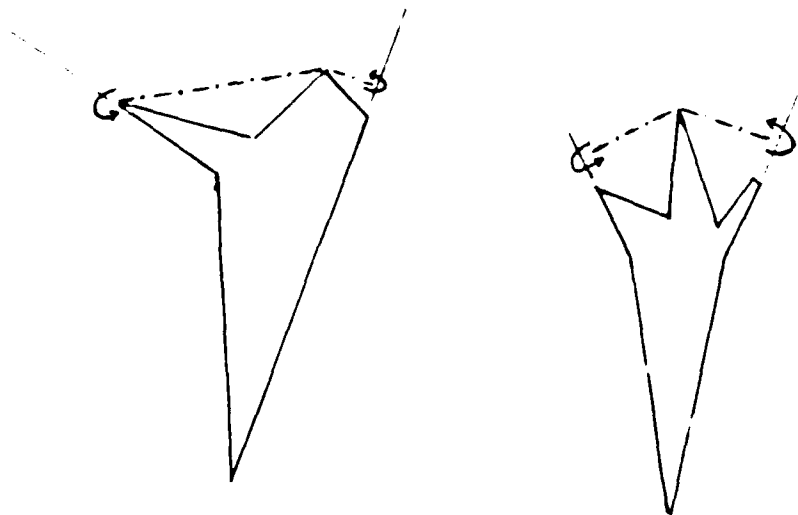


Figure 11. Shortest external routes for star polygons.